Fractal Properties of Quasicrystalline Penrose Tiling Synthesis

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The synthesis of decagonal Penrose tiling by a pair of decagons – starlike and dorsal is considered in this paper. The synthesis procedure is realized according to morphogenetic growth model. It is shown that procedure which controls the synthesis process is just chaotic, but not stochastic. The fractal properties of Penrose tiling synthesis procedure are studied. It is revealed that all characteristics via the morphogenetic fronts have oscillatory-wave behavior.

Keywords: Decagonal Penrose tiling, Starlike decagon, Dorsal decagon, Morphogenesis, Entropy, Chaos, Fractal.

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1. INTRODUCTION

In the basis of Penrose tiling synthesis the three alphabet levels, which form the hierarchical system, lay (Fig. 1). The first level is formed by two "golden" Robinson triangles, this level is the basic or symbol one. The second level is formed by two "golden" rhombuses; this level is syllable one. The third alphabet level is formed by two types of decagons, starlike (*S*-) and dorsal (*D*-). This, third alphabet level {*S, D*} is naturally considered as phraseological, since each decagon is aggregated by 10 syllables, or rhombuses. Thereby we have three levels of dual alphabets which form the complete system.

The paving procedure of R_2 plane by a pair of golden rhombuses is a part of nonrecursive mathematics as R. Penrose notice [1, 2]. For example papers [3, 4] contain a quite extensive research for problems of central cartwheel defect and dead surfaces in the paving procedure. These problems may be avoided, and synthesis procedure may be easier if to work with decagons which correspond to the third alphabet level. Here certain nontrivial rules for the decagons conjunction present, which permit some overlappings or defects (Fig. 2).

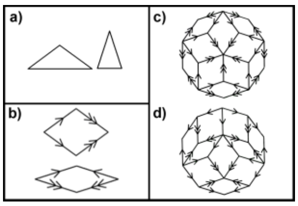

Fig. 1 – Three levels of Penrose tiling synthesis

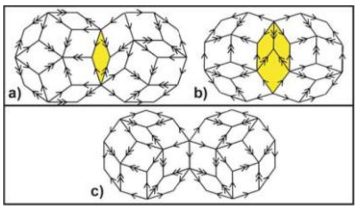

Fig. 2 – The conjunction rules for starlike (*S*-) and dorsal (*D*-) decagons. (a-c) The types of mutual conjunctions

2. FORMALISM

For morphogenetic fronts analysis we employ the theory of enumeration [5]. Any enumerating polynomial (EP) has the standard form:

$$EP_i\left(\hat{P}\right) = \sum_{k_i=1}^{4} T(y_{k_i}) \cdot y_{k_i} \ ,$$

where $T(y_{k_i})$ – the coefficients of EP, defining the bushes number with k_i – the branching coefficient; y_{k_i} – the bushes designation.

It is seen that each EP has two freedom degrees, one of them describes the tangential "phrase" length, and the second describes the number of "phrases" with specified length, which are contained in the current motive.

The probabilistic EP (PEP) might be turned from absolute form by an ordinary normalization on the each motive of hierarchy:

$$NEP_i\left(\hat{P}\right) = \sum_{k_i=1}^{4} t(y_{k_i}) \cdot y_{k_i} \ ,$$

where $t(y_{k_i}) = T(y_{k_i}) \big/ \sum_{k_i=1}^{4} T(y_{k_i})$ – the normalized coefficients for y_{k_i} .

Due to normalization procedure this expression receives a new significance: now it might be considered as a probability distribution or the statistics of "phrases" by the length. This statistic gives an opportunity to find an average length on each level, and further the entropy of length:

$$\bar{k}_i = \sum_{k_i=1}^{4} k_i \cdot t(y_{k_i}) \Rightarrow \ln(\bar{k}_i) = S_i(\bar{k}_i) \ .$$

According to information-dynamical method [6-11], one may calculate the fractal dimension estimate. Firstly the entropy dependence in the Radon function form – $H\left[S\left(\bar{k}\right)\right]$ should be calculated. The Radon function in turn allows existence of the derivative in terms of Radon-Nicodym [12], which linearized form in our

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opinion, is fractal dimension value:

$$\frac{\partial H\left[S(\bar{k})\right]}{\partial S(\bar{k})}=\frac{H_i\left(t\left(y_k\right)\right)}{S_i\left(\bar{k}\right)}=d_{fr}\left(i\right)\Rightarrow\left\langle \bar{d}_{fr}\left(\hat{P}\right)\right\rangle .$$

Last expression gives useful information-theoretical interpretation for the fractal dimension. In the denominator, the entropy for average length $S_i\left(\bar{k}\right)$ stands, which in turn corresponds to the diversity portion, carried out by the second alphabet symbol, on average. Therefore, the numerator $H_i\left(t\left(y_k\right)\right)$ corresponds to the diversity portion which depends on the "phrase" repetition factor. These entropies ratio that is an entropic diversity density.

3. RESULTS AND DISCUSSION

We consider that decagonal Penrose tiling synthesis procedure is realized with respect to the morphogenetic growth model. Our interpretation of morphogenesis is quite close to the terms of monograph [13]. In the basis of morphogenesis always lays certain seed gene, in our case *S*-decagon. The reduplication mechanism itself is appeared to be as systems of a wavy fronts, which are consist of {*S, D*} alphabet symbols in the special succession and order (Fig. 3).

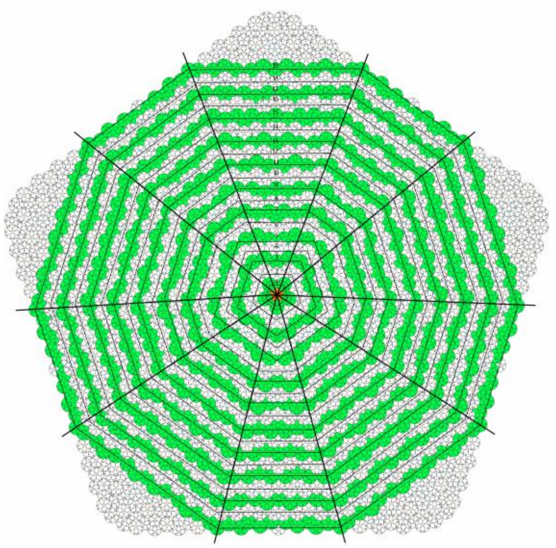

Fig. 3 – The result of decagonal Penrose tiling synthesis procedure with highlighted morphogenetic fronts and angular sectors

It is clearly seen that in the result of decagonal covering we at the same time indirectly have solved the synthesis task on the rhombuses level. The total number of decagons for 20 morphogenetic fronts yields 1245. The decagons probabilities in such a sample are $p(S) = 0.309\ldots$, $p(D) = 0.691\ldots$

Figure 4 shows the dependence of *S*- and *D*-decagons numbers ratio via hierarchy levels, and also the ratio $[n(D) - n(S)] / [n(D) + n(S)]$. The visual analysis points to the quasistochastic periodicity. Here we also have analyzed the flow of nulls for these trends and got the estimates for zero-repetition intervals $\overline{T_0}\left(n(D)/n(S)\right)=1.42\pm 45\%$,

$\overline{T_0}\left(\left[n(D)-n(S)\right]/\left[n(D)+n(S)\right]\right)=1.39\pm 36\%$. Making an integration for these values, we obtain a twice average estimate for the zero-repetition interval $\left\langle \bar{T}_{00}\right\rangle =1.41\pm 40\%$.

Figure 4a, b shows the significant variability, which in our opinion is conditioned by a high level of chaoticity [14, 15], which originated from the geometry of the decagon conjunctions. There is some justification, for last assertion. If we worked with primary alphabet in a random manner, according to its probabilistic structure, we would come to non-zero probability $p(SS) \sim 9$ %. But this is prohibited by synthesis logic of decagonal tiling (Fig. 2). This means that the procedure of stochastic synthesis of tiling prohibited principally. On the other hand, we have a significant variability of zero-repetition interval estimates. Then there is only way to recognize that procedure which controls the synthesis process is just chaotic [10, 11].

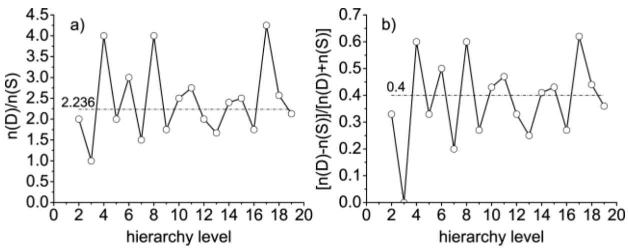

Fig. 4 – The dependences of *S*- and *D*-decagons numbers ratio (a), and swing $[n(D) - n(S)] / [n(D) + n(S)]$ (b) via hierarchy levels

From the traditional crystallography point of view, the behavior of Fig. 4a, b dependencies allow describe the process of radial morphogenesis via fractional translations with substantially fluctuate irrational period. It is remarkable that in the strictly deterministic morphogenesis procedure the stochastic is absolutely absent.

From Figure 3 substantially different role of *S*- and *D*-decagons in tangential "phrases" is seen. Each "phrase" always starts with an *S*-decagon, which means the unnecessary of separation signs. All the "phrases" can be described by the $\{SD_{i\le 4}\}$ expression.

The self-similar character of "phrases" allows employing the theory of enumeration [5]. The normalized EP might be turned from absolute form by an ordinary normalization on the each motive of hierarchy. Due to normalization procedure the PEP receives a new significance: now it might be considered as a probability distribution or the statistics. This statistic gives an opportunity to find an average "phrase" length on each morphogenetic front, and further the entropy of length.

Now one may consider the second freedom degree of enumerating polynomials in the normalized (or probabilistic) form – the probabilistic coefficients, and introduce the entropy function for them. Figure 5 once again demonstrates the oscillation-wave feature presence. Probably this feature will be the universal for whole quasicrystalline symmetries. The averaged entropy value for the stationary stage yields ~0.80.

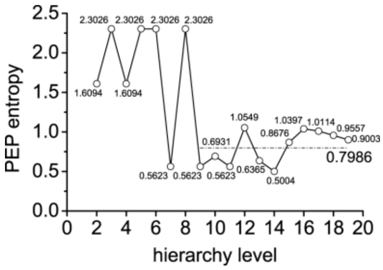

Fig. 5 – The entropy values for coefficients of probabilistic enumerating polynomials via morphogenetic fronts

Further, according to information-dynamical method [6-11], one may calculate the fractal dimension estimate for the decagonal Penrose tiling synthesis procedure.

Firstly the entropy dependence in the Radon function form – $H\left[S(\bar{k})\right]$ should be calculated. The Radon function in turn allows existence of the derivative in terms of Radon-Nicodym [12], which linearized form in our opinion, is fractal dimension value (Fig. 6).

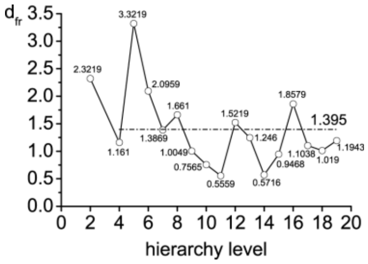

Fig. 6 – The fractal dimension values via morphogenetic fronts

The dependency on the Figure 6 has essentially oscillation-wavy character and the first average value yields $\bar{d}_{fr1}\left(\hat{P}\right)=1.395...$. If it is required to make an elimination of the global spike on the third level, then the average value will be $\bar{d}_{fr2}\left(\hat{P}\right)=1.275...$ And the general average value for two these values will be $\left\langle \bar{d}_{fr}\left(\hat{P}\right)\right\rangle = 1.33...$

One interesting conclusion here is that the morphogenetic front is not linear at all. In spite of its monolinear Turing construction it possesses the fractal dimension. Thereby even the tangential component of Penrose tiling is "ultralinear".

For the further consideration of fractality feature we suggest make a research of the properties of chaotic attractor if such exists in the synthesis procedure. According to the dynamical chaos conception [14 – 16], it is necessary to construct the corresponding phase space and to carry out a typical analysis of phase trajectory for the purpose of attractor type identification.

Usually the phase space is based on the generalized coordinate-impulse axes with subject to freedom degree number. In our opinion here the Radon's function methodology [12] must be employed with the purpose of forming of entropy functionals from these freedom degrees. Just these entropies we declare to be generalized coordinates in the Radon's terms. Figure 7 shows such a phase trajectory.

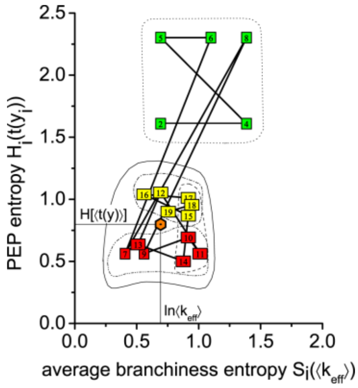

Fig. 7 – Phase trajectory of Penrose tiling synthesis in the axes $\left[H_i\left(t\left(y_k\right)\right);S_i\left(\bar{k}\right)\right]$ which the Radon methodology suggests

Fig. 7 also shows that attractive set has two centers. The upper right center corresponds to the nonstationary mode in the initial morphogenesis stage. Further, morphogenetic growth was gone, the lower left cluster appeared. It is a compact set as any attractor is supposed to be.

The same $H_i\left(t\left(y_k\right)\right)/S_i\left(\bar{k}\right)$ ratio is typical for both sets and points to some information-theoretical invariant, which is no other than fractal dimension in the methodology of entropic [17] diversity density of probabilistic enumerating polynomials.

This attractive set possesses the specific subtle structure consisted of three sub-clusters (Figure 7). It is notable that "horizontal" clusters possess the dorsal symmetry axis, which passes over point ln 2. The whole cluster along the $H_i\left(t\left(y_k\right)\right)$-axis lays in the range 0.5÷1.05, and along $S_i\left(\bar{k}\right)$-axis in the range 0.336 (K_{min} = 1.4) ÷ 0.916 (K_{mod} = 2.5); K_{max} = 2.745. The remarkable fact is a barycenter presence in the point with location (ln 2; 0.8).

The barycenter concept is quite unobvious, and not all attractors possess it. The identification of attractor type means the determination of phase trajectory behavior character in the barycenter neighborhood. To this effect we have calculated the distances between points of phase trajectory and the barycenter in the Euclidean topology (Fig. 8).

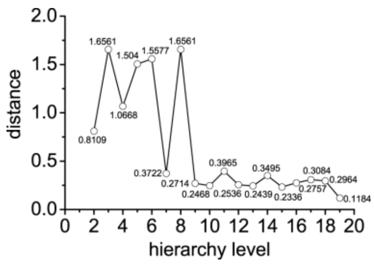

Fig. 8 – The graph of distances between phase Radon's trajectory points and attractor's barycenter in the Euclidean topology

Figure 8 clearly shows three behavioral stages of current dependency. The first stage is a "2-6" levels ascending region, corresponding to the moving phase trajectory away from the barycenter. This diverging branch has an average logarithmic tendency.

The second stage which corresponds to levels "7-8" reveals the sharp spontaneous fluctuating decreasing of current dependency, which means the high fluctuative attraction to the barycenter on these fronts of morphogenesis process. Just this stage and this effect we call the "phase transition". Let us explain what concepts underlie this effect. One may turn back to Figure 3 and thoroughly analyze it. This pattern reveals that earlier morphogenetic fronts "1-5" possess pentagonal symmetry and further peripherad the symmetry type sharply changing to the decagonal symmetry. One may easily determine the transition rules:

a) Each pentagon vertex is translated into the new edge of decagon;

b) Each pentagon edge induces new parallel edge of decagon.

These factors allow make a conclusion that we have phase transition with spontaneous symmetry breaking $5 \rightarrow 10$. It is a purely Landau symmetry-type phase transition.

The third stage "9-19 $\rightarrow \infty$" corresponds to the oscillation-wave behavior of Radon trajectory in the phase space. All these levels correspond to the stable dense cluster with focus attractor. But from the other side the oscillations of "9-19" levels have not damping tendency, and consequently it is the cyclic attractor.

Then this oscillation-wave behavior is equivalent to the astable, autowave process to which the intense oscillations of earlier stages are tended by focus scenario. Actually the autoregime, rhythm is established. So just such a regime characterizes this morphogenesis process.

In the ordinary coordination space, morphogenetic fronts are growing (reduplicating) from seed gene to infinity. From the attractors theory point of view, the growing process from the centre to the periphery is gone since the infinite horizon is just an attractor, to which all the morphogenetic fronts are tending. If to consider this process from the statistical thermodynamics point of view, than in spite of quasiwave morphogenesis character it has some specific features of thermal relaxation processes as it would be unexpected.

It must be once again noticed that before phase transition system is undergone catastrophical fluctuations (on the "7-8" levels) and just only after that the compact stable cluster is forming. Also this phase transition is characterized by invariant (in general), which is no other than a fractal dimension property.

Throughout the whole article are repeatedly faced with essential quasiperiodicity of morphogenesis process. Also, we have discussed that the oscillation-wave behaviour of different characteristics of the Penrose tiling is its inherent feature.

Such an originality of this approach of fractality and attractor features research of morphogenesis is conditioned just by the choice of phase space in the Radon's coordinates. It is no need to calculate the Poincare cross-sections [15] or to research the Lyapunov exponents spectrum [15].

This approach for analyzing of complex hierarchical multidimensional systems is based on the interpretation of fractality feature just as it prescribes the Radon-Nicodym derivative. This derivative is a ratio of "phrases" repetition coefficient's entropy to the "phrases" length coefficient's entropy in the polynomial representation of complex hierarchical systems, such as a Penrose tiling.

4. CONCLUSION

We have made a synthesis of a decagonal Penrose tiling (by the third alphabet level), which has been revealed as having a fractal properties with value of fractal dimension ~ 1,3. In our opinion this inherent fractality feature is the necessary condition for the decagonal Penrose tiling synthesis procedure itself.

The remarkable point is that fractal dimension by Fig. 6 is correlated with fractal dimension of chaotic attractor (Fig. 6), describing the morphogenesis procedure in phase space with $\left[H_i\left(t\left(y_k\right)\right);S_i\left(\bar{k}\right)\right]$ axes. Thereby in spite of visual linearity of morphogenetic fronts, in truth it is "ultralinear". The subtle structure of chaotic attractor consisted of three stages confirms this fact.

The first stage (fronts "1-6") is a nonstationary part of morphogenesis process, with fivefold fronts symmetry. The second stage (fronts "7-8") is characterized by high fluctuations, which means Landau phase transition with symmetry breaking from pentagonal to decagonal $5 \rightarrow 10$. The third stage ("9-19 $\rightarrow \infty$" levels) is an established one.

The cyclic focus attractor on the phase space which corresponds to the autowave mode is appeared. Thereby the dynamics of morphogenetic evolution in the phase space contains the thermodynamic, fluctuating, and autowave modes. *Eo ipso* the decagonal Penrose tiling synthesis procedure is an chaotic and fractal one.

REFERENCES

1. R. Penrose, *Bull. Inst. Math. Applic.* **10**, 266 (1974).
2. R. Penrose, *The Emperor's New Mind* (Oxford University Press: Oxford: 1989).
3. G.Y. Onoda, P.J. Steinhardt, D.P. DiVincenzo, J.E.S. Socolar, *Phys. Rev. Lett.* **60**, 2653 (1988).
4. H.-C. Jeong, *Phys. Rev. Lett.* **98**, 135501 (2007).
5. F. Harari, E.M. Palmer, *Graphical Enumeration* (Academic Press: New York: 1973).
6. V.V. Yudin, T.A. Pisarenko, E.A. Lyubchenko, *Informodynamics of Network Structures* (Far-East Federal University: Vladovostok: 2003) [in Russian].
7. V.V. Yudin, P.L. Titov, A.N. Mikhalyuk, *Bull. RAS* **73**, 1269 (2009).
8. V.V. Yudin, P.L. Titov, A.N. Mikhalyuk, *Theor. Math. Phys.* **164**, 905 (2010).
9. A. Mihalyuk, P. Titov, V. Yudin, *Physica A* **389**, 4127 (2010).
10. V.V. Yudin, Yu.A. Karygina, *Crystallogr. Rep.* **46**, 922 (2001).
11. V.V. Yudin, T.A. Pisarenko, E.A. Lyubchenko, O.A. Chudnova, Yu.A. Karygina, *Crystallogr. Rep.* **47**, 189 (2002).

12. R.E. Edwards, *Functional Analysis. Theory and Applications* (Holt, Rinehart and Winston: New York: 1965).
13. W. Ebeling, A. Engel, R. Feistel, *Physik der Evolutionsprozesse* (Akademie-Verlag: Berlin: 1990).
14. G.M. Zaslavsky, R.Z. Sagdeev, D.A. Usikov, *Usp. Fiz.* **31**, 887 (1988).
15. T.S. Akhromeeva, S.P. Kurdyumov, G.G. Malinetskiy, A.A. Samarkiy, *Nonstationary Structures and Diffusional Chaos* (Nauka: Moscow: 1992) [in Russian].
16. R.M. Crownover, *Introduction to Fractals and Chaos* (Jones and Bartlett Publishers: Boston: 1995).
17. N.F.G. Martin, J.W. England, *Mathematical Theory of Entropy* (Addison-Wesley Publishing Company: London: 1981).